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Correlations between Rare Events for Gaussian Stochastic Processes with Long-Term Memory

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(Received 15 December 2025; accepted 8 June 2026; published 10 August 2026)

Rare events refer to qualitatively unlikely events whose realization can nevertheless have important consequences. Typically, the prediction of the kinetics of these events relies on Arrhenius laws, with exponentially distributed waiting times, and no correlations between successive occurrences. However, this description breaks down in the presence of long-term memory, as has been observed in the contexts of geophysical time series or protein dynamics. So far, existing analytical approaches do not quantify the correlations between rare events due to long-term memory. Here, for non-Markovian Gaussian processes, we determine analytically the impact of long-term memory on the distribution of first and second passage times to a rarely reached threshold, using a perturbation approach. This distribution is nonexponential, thus going beyond the Arrhenius paradigm. We obtain an explicit expression for the covariance between the first and second passage times, and we predict how the mean time to the next extreme event depends on the previous passage time, illustrating the phenomenon of clustering of extreme events. These analytical results, validated through extensive stochastic simulations, shed lights on the strong correlation between successive occurrences of extreme events due to long-term memory.

DOI: 10.1103/w2kv-jkvy

Many physical and chemical processes are controlled by "rare" events, referring to qualitatively unlikely events that can nevertheless have important consequences when they are realized [1–3]. Such events appear in various areas of science, as exemplified by the formation or rupture of bonds [1,4–7] in chemical physics, molecular motor dynamics [8,9], nucleation events, stock market crashes [10], and climate [11] or population [12,13] dynamics.

A key question about rare events is to characterize how often they occur. The paradigm describing this occurrence kinetics is given by Arrhenius (or Kramers) laws, in which (i) the distribution of times to reach a rare event (waiting times) is exponential and does not depend on initial conditions, and (ii) the mean waiting time is exponentially large with the energy barrier that has to be overcome to reach the target configuration [1] (or the pseudopotential barrier for nonequilibrium systems [14–17]). Arrhenius laws are derived in the weak noise limit by analyzing the dynamics at the top of the (pseudo)potential barrier. In this limit, waiting times usually become larger than all relaxation times of the dynamics, and are thus independent of initial conditions. Also, for the same reason, there are no correlations between successive occurrences of rare events.

However, these properties may be questioned for stochastic processes $x(t)$ with long-term memory, for which correlation functions decay as a power law rather than exponentially, i.e., which satisfy

$$\phi(\tau) \equiv \lim_{t\to\infty} \frac{\langle x(t)x(t+\tau)\rangle}{\langle x^2(t)\rangle} \underset{\tau\to\infty}{\simeq} \frac{A}{\tau^{\alpha}}, \tag{1}$$

where $A \neq 0$, $0 < \alpha < 1$, and $\langle x(t)\rangle = 0$ by convention. The above property, hereafter referred to as "long-term memory property" [18,19], typically arises for processes resulting from the evolution of an infinite number of degrees of freedom with broadly distributed timescales, for example, in the context of the dynamics of polymers [20], proteins [21–24], interfaces [25], critical Ising models [26], active baths [27], neurons [28], earthquakes [29], or rainfalls [30]. For long-term memory processes, the relaxation time to reach a stationary state is infinite, and thus cannot be smaller than the Arrhenius time. This questions the validity of the oversight of initial conditions and of the independence between successive occurrences of extreme events. Even though for Gaussian processes with long-term memory, mathematical results indicate that Arrhenius laws are still valid [31,32], deviations from the Arrhenius paradigm are observed in different contexts. First, the phenomenon of "clustering of extreme events" has been identified and is relevant for various geophysical time series (rainfalls, temperatures, etc.) [19,33]. Second, long-term memory may be involved in the nonexponential distribution of barrier crossing times [34,35] which are commonly observed in protein dynamics [36–40].

Although the effect of memory on rare event kinetics has been investigated recently [5,41–56], most analytical approaches overlook long-term memory. Existing theories,

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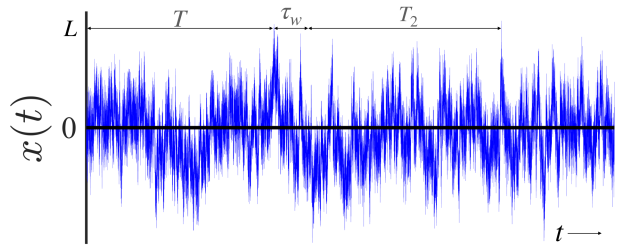

FIG. 1. Sketch of the problem. For a nonsmoooth Gaussian stochastic process $x(t)$ with long-term memory, what is the distribution of the first passage time T to a rarely reached threshold L ? More generally, what are the correlations between T and the second passage time, T_2 (measured after a lag time τ_w) ? The shown trajectory corresponds to a process $x(t)$ satisfying Eq. (2), with $\alpha = 1/2$.

including long-term memory, rely on the use of the generalized Fokker-Planck equation [49] (which is inconsistent with simulations [57,58]), the Wilemski-Fixman [50] or independent return intervals [18] approximations (which neglect a number of memory effects), including phenomenological parameters to describe temporal fluctuations of energy barriers (dynamic disorder) [35,59], or approximately dividing between slow and fast variables [5]. Recently, an approach that analyzed the paths after the rare event was proposed but predicted only the mean waiting times [60]. Hence, the analytical prediction of their distribution, as well as of the correlations between successive rare events, remains unresolved.

Here, we address this question in the case of Gaussian processes with long-term memory (Fig. 1). We develop a theory that analytically predicts how the distribution of first passage times to a rarely reached threshold deviates from an exponential distribution due to long-term memory. The theory is then extended to obtain simple and explicit formulas describing the correlations between the first and second passage times, in excellent agreement with extensive simulations, identifying the key parameters controlling the dependence of the future rare event times on past ones.

Model—We first consider the position $x(t)$ of an overdamped particle in a harmonic potential (with stiffness k), submitted to noninstantaneous friction with kernel $K(t)$, at temperature $\mathcal{T}$, obeying the generalized Langevin equation

$$\int_0^t dt' K(t-t')\dot{x}(t') = -kx(t) + \xi(t), \tag{2}$$

where the centered Gaussian noise $\xi(t)$ is correlated as $\langle \xi(t)\xi(t')\rangle = k_B\mathcal{T}K(|t-t'|)$. We focus on scale invariant kernels $K(t) = K_\alpha/[\Gamma(1-\alpha)t^\alpha]$, so that $x(t)$ satisfies the long-term memory property (1) with $A = K_\alpha/[\Gamma(1-\alpha)k]$. With these hypotheses, $x(t)$ describes a number of physical processes, such as a tracer particle in a viscoelastic fluid [61–63], a tagged monomer of a long polymer [20,64,65], or the distance between two protein residues as experimentally observed [22]. Our aim is to characterize the statistics of the first passage time (FPT), called T, of $x(t)$ to a threshold value L that is anomalously high ($L \gg l$, with $l = \langle x^2\rangle_s^{1/2} = \sqrt{k_B\mathcal{T}/k}$). Reaching L requires to overcome the energy barrier $\Delta E = kL^2/2 \gg k_B\mathcal{T}$ and is thus rare. We will also characterize the second passage time T_2, defined as the time to obtain a second threshold crossing. Although we consider here a passive model, all our results will be generalized to active ones (see below).

Since Kramers escape times usually depend on the dynamics near the top of the potential barrier, here identified as $x = L$, we first identify the characteristic timescale t^* and length scales l^* of this dynamics. From dimensional analysis, we identify $l^* = k_B\mathcal{T}/(kL)$, where kL is the slope of the potential at the target position L. At short times, the fluctuations of $x(t)$ read $\mathrm{var}[x(t)]_{x(0)=L} \simeq \kappa t^{2H}$ with $H = \alpha/2$ and $\kappa = 2k_B\mathcal{T}/[\Gamma(1+\alpha)K_\alpha]$. This implies that $x(t)$ is nonsmooth, i.e., displaying nondifferentiable trajectories (like Brownian motion) [25]. Defining t^* as the time at which the spatial fluctuations of $x(t)$ become of the order of l^* leads to $t^* = (l^*/\sqrt{\kappa})^{1/H}$.

First passage to a threshold: deviation from Arrhenius law—Our starting point is the following generalized renewal equation [66]:

$$p(L,t) = \int_0^t d\tau F(t-\tau)p(L,t|T=t-\tau), \tag{3}$$

where $t > 0$, $p(x,t)$ is the probability density function (PDF) of $x(t)$, $F(t)$ is the PDF of first passage times, and $p(L,t|T=t')$ is the probability density of observing $x(t) = L$ given that the FPT is t', if the process is allowed to continue after the first passage. We assume that x starts with a stationary distribution, so that $p(x,t) = p_s(x) = e^{-kx^2/(2k_B\mathcal{T})}/\sqrt{2\pi l^2}$. The exact Eq. (3) is obtained by partitioning the probability to observe $x(t) = L$ over the values of the FPT, which is necessarily smaller than t due to the nonsmooth property of trajectories.

Our approach consists of exploiting the fact that the first passage kinetics results from the contribution of two widely separated timescales. The first relevant timescale t^* characterizes the dynamics near the threshold. A second important timescale is the mean FPT in the weak noise limit, called T_{RE}, which grows as $T_{\mathrm{RE}} \propto e^{\Delta E/k_B\mathcal{T}}$ [60] and is thus much larger than t^*. Since $F(t)$ varies at the scale T_{RE}, it is essential to understand how the propagator behaves at this timescale. A key remark is that, following an event where $x(0) = L$ (not for the first time), the relaxation to the steady state is algebraic,

$$p(L,t|L,0) \underset{t\to\infty}{\simeq} p_s(L)\left(1 + \frac{L^2A}{l^2t^\alpha}\right) \simeq p_s(L)\left(1 + \frac{\varepsilon}{\bar{t}^\alpha}\right). \tag{4}$$

This expression can be derived from the known solutions of Eq. (2). In the second equality, we have used the rescaled time $\bar{t} = t/T_{\mathrm{RE}}$, and this has led us to introduce the parameter $\varepsilon \equiv AL^2/(l^2 T_{\mathrm{RE}}^{\alpha})$, which will be a key small parameter of our theory. Our strategy is then to expand the relevant quantities in powers of ε: we set $F(t) \simeq T_{\mathrm{RE}}^{-1}[F_0(\bar{t}) + \varepsilon F_1(\bar{t}) + \cdots]$, while for the propagator after the FPT, we use

$$p(L, t|T = t - \tau) \simeq p_s(L)[1 + \varepsilon R(\bar{\tau}, \bar{t} - \bar{\tau}) + \cdots], \quad (5)$$

with R a dimensionless function and $\bar{\tau} = \tau/T_{\mathrm{RE}}$. In turn, if the time τ after the FPT is of the order of t^*, dimensional analysis indicates that $p(L, t|T = t - \tau) \simeq q^*(\tau/t^*)/l^*$, where q^* is a dimensionless scaling function.

Next, we decompose the integral over τ in Eq. (3) into two parts: one for $\tau \sim t^*$ (where the propagator is proportional to q^*), and another part for $\tau \sim T_{\mathrm{RE}}$ [where we use Eq. (5)]. Using this procedure, we obtain

$$\begin{aligned} p_s(L) = {} & \frac{F_0(\bar{t}) + \varepsilon F_1(\bar{t})}{l^* T_{\mathrm{RE}}} t^* c_H + p_s(L) \\ & \times \int_0^{\bar{t}} d\bar{\tau} \left[F_0(\bar{t} - \bar{\tau}) + \varepsilon F_1(\bar{t} - \bar{\tau})\right] \\ & \times \left[1 + \varepsilon R(\bar{\tau}, \bar{t} - \bar{\tau})\right], \end{aligned} \quad (6)$$

where $c_H = \int_0^\infty dv q^*(v)$ is a numerical constant depending only on H. For $\varepsilon = 0$, the above integral equation for F_0 can be readily solved, yielding

$$F_0(\bar{t}) = e^{-\bar{t}}, \qquad T_{\mathrm{RE}} = \frac{t^* c_H}{l^* p_s(L)}. \quad (7)$$

As expected, we obtain the exponential distribution of the Arrhenius paradigm, with a mean time that is exponentially large with the energy barrier, $T_{\mathrm{RE}} \propto 1/p_s(L) \propto e^{\Delta E/k_B \mathcal{T}}$. The above formula is consistent with the mathematical results of Pickands [31,32] and previous results [52].

To obtain the effects of long-term memory, we analyze (6) at order ε, obtaining a relation between R and F_1,

$$F_1(\bar{t}) + \int_0^{\bar{t}} d\bar{\tau} \left[F_1(\bar{t} - \bar{\tau}) + e^{-\bar{t} + \bar{\tau}} R(\bar{\tau}, \bar{t} - \bar{\tau})\right] = 0. \quad (8)$$

To obtain an equation for R, we write a generalized version of the renewal equation (3), for any $t > 0$, and any set $0 < t_1 < t_2 < \ldots < t_m$ and $x_1, x_2, \ldots, x_m$,

$$\begin{aligned} & p(L, t; x_1, t + t_1; \ldots; x_m, t + t_m) \\ & = \int_0^t dt' F(t') p(L, t; x_1, t + t_1; \ldots; x_m, t + t_m | T = t'). \end{aligned} \quad (9)$$

To close the system, we assume that the trajectories after the first passage $T = t$ follow Gaussian statistics, with a covariance equal to the covariance of trajectories conditional to $x(t) = L$ (not for the first time). The function R is in fact the rescaled mean of the trajectories in the future of the FPT, and an equation for R can be found by investigating the above equation for $m = 1$ with the above mentioned method of taking into account all timescales and collecting terms of order ε (see End Matter). The resulting equations for R and F_1 can be explicitly solved, leading to

$$F_1(\bar{t}) = \bar{t}^{1-\alpha} \frac{\bar{t} - 2 + \alpha}{(2 - \alpha)(1 - \alpha)} e^{-\bar{t}}. \quad (10)$$

This formula provides the first corrections of F to the exponential behavior due to long-term memory, which take the form of a universal function depending only on α. It was obtained in a perturbation approach based on the separation of timescales $t^* \ll T_{\mathrm{RE}}$ in the rare event limit, assuming Gaussian statistics after the first passage time $t = T$ with a covariance approximated by the covariance of initial trajectories conditional to $x(t) = L$. This kind of approximations was found to be very accurate in other contexts [52,66]. Here, they can be self-consistently justified by noting that, at order ε, Eq. (9) is satisfied for all m and all x_i as soon as it is satisfied for $m = 1$, suggesting that they are in fact exact at order ε at the timescale T_{RE} [see Supplemental Material (SM) [67]]. Using the above expressions, one obtains for the moments $\langle T^n \rangle = T_{\mathrm{RE}}^n [n! + \varepsilon n \Gamma(n + 2 - \alpha)/(2 - 3\alpha + \alpha^2)]$, which is compatible with the results of Ref. [60] (restricted to $n = 1$).

To test our approach, we have generated stochastic trajectories $x(t)$ satisfying Eq. (2) with stationary initial conditions by using a modified version of the circulant matrix algorithm [68,69], as done in Ref. [60]. Trajectories are sampled exactly at times $t_n = n dt$, with dt the time step, and we measured the distribution of FPT; see Fig. 2(a). Deviations from exponential distributions are clearly observed, and correctly captured by our theory, at least when they are small (as expected for a perturbation calculation). To further quantify the deviation from exponential behavior, we define $d_e = \langle T^2 \rangle / \langle T \rangle^2 - 2$, so that $d_e = 0$ for exponentially distributed T. In our theory,

$$d_e \simeq 2\Gamma(2 - \alpha) \frac{L^2 A}{l^2 \langle T \rangle^\alpha}. \quad (11)$$

This formula is validated numerically for various values of α in Fig. 2(b), indicating that our theory is able to quantify the first corrections to exponential behavior of the FPT distribution due to long-term memory.

Correlations between successive passage times—We now ask the question of the second passage time T_2. Actually, many threshold crossings are expected at times $t \sim t^*$ after the first passage, and for a nonsmooth process the next-passage time is formally zero. Instead of looking at these short-time recrossing events, we focus on the more relevant quantity, which is the time to come back to the threshold once the particle has moved away from it. Hence,

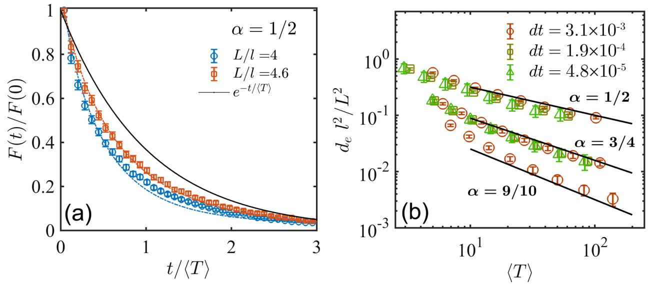

FIG. 2. (a) The distribution $F(t)$ of FPTs, renormalized by $F(0)$ for better readability, is plotted for the processes described by Eq. (2) for $\alpha = 1/2$. Symbols, simulations; dashed lines, prediction $F(t)/F(0) = e^{-v}\{1+\varepsilon[v^{1-\alpha}(-2+\alpha+v) - \Gamma(3-\alpha)v]/[(2-\alpha)(1-\alpha)]\}$ with $v = t/\langle T\rangle$ and $\varepsilon = AL^2/(l^2\langle T\rangle^\alpha)$, which is compatible at order ε with Eq. (10). (b) Quantity $d_e l^2/L^2$, measuring the deviation of $F(t)$ to the exponential distribution, for the same process, with $\alpha = 1/2, 3/4$ and $9/10$. Symbols, simulations, with the value of the used time step in legend; lines, prediction (11). Units were chosen so that $K_\alpha = k = k_B\mathcal{T} = 1$.

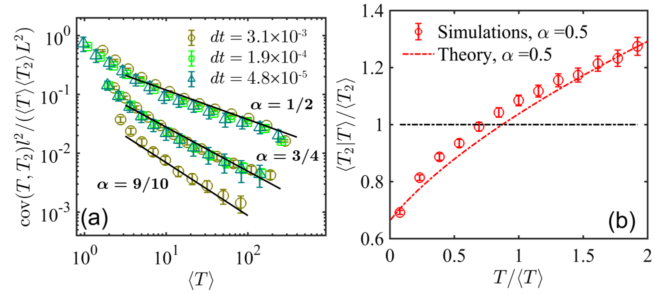

FIG. 3. (a) Normalized covariance between the first and second passage times T and T_2 for the passive process described by the generalized Langevin equation (2). Symbols, simulations; lines, analytical prediction (12). Here, $\tau_w = 1$. (b) Mean second passage time T_2, conditional on T, against $T/\langle T\rangle$ for the same process. Symbols, simulation results; line, prediction (13), with $\varepsilon = L^2A/(l^2\langle T\rangle^\alpha)$; parameters, $\tau_w = 1$, $L/l = 3.6$, $dt = 1.91\times 10^{-4}$. Units are such that $K_\alpha = k = k_B\mathcal{T} = 1$.

we introduce a lag time τ_w and we define T_2 so that, after $T+\tau_w$, the threshold is hit at $T+\tau_w+T_2$ for the first time, and T_2 does not depend on τ_w provided $t^* \ll \tau_w \ll T_{\mathrm{RE}}$ (see SM [67]).

To predict the statistics of T_2 conditional to a fixed value of T, we see T_2 as a first passage problem for the process in the future of the real first passage time, for which we know the mean and the covariance as outputs from the previous analysis. We thus adapt the same procedure to take into account these complicated initial conditions, leading to an analytical formula of the conditional distribution of T_2 given T. Since we know the distribution of T, this leads to determining the full joint distribution of T_2, T (see End Matter). In particular, for their rescaled covariance, we find

$$\frac{\mathrm{cov}(T,T_2)}{\langle T\rangle\langle T_2\rangle} \simeq \frac{AL^2}{2l^2\langle T\rangle^\alpha}\Gamma(3-\alpha). \qquad (12)$$

This is the main result of the present Letter. Of note, although our theory is based on a perturbative expansion in powers of ε, here the perturbation results for the covariance between T, T_2 is actually the leading order term because in the Arrhenius paradigm there are no correlations between successive rare events. It is validated by comparing with simulations for various values of α; see Fig. 3(a). Equation (12) indicates that T and T_2 are positively correlated. Physically, this can be understood as follows: if T is unusually large, it is likely that, due to the other degrees of freedom, the energy barrier to reach the threshold is higher than usual, and one can expect that the next passage will tend to be higher than usual because it also faces this higher energy barrier, which, due to the long-term memory, will not relax fast to its average value. Our theory is of interest as it provides the information about the correlations between successive events without assuming *a priori* a dynamics for this barrier, leading to a simple law depending only on a restricted number of parameters: A (appearing in the large time behavior of the correlation function), the threshold L, the mean time $\langle T\rangle$, and the stationary amplitude of fluctuations $l^2 = \langle x^2(t)\rangle_s$.

To further illustrate the correlations between T, T_2, we consider the expected time to the second passage given the first passage, for which we obtain

$$\frac{\langle T_2|T=\tau\rangle}{\langle T_2\rangle} \simeq 1 + \varepsilon\frac{e^{\bar\tau}\Gamma(2-\alpha,\bar\tau) - \Gamma(3-\alpha)}{1-\alpha}, \qquad (13)$$

where $\bar\tau = \tau/\langle T\rangle$. This expression is in good agreement with numerical simulations; see Fig. 3(b). It also shows that the time to the next event can be expected to be higher than its global average if the previous passage time was observed to be unusually large. The positive correlations between successive events [Eqs. (12) and (13)], together with the nonexponential distribution of waiting times [Eq. (10)] are characteristic features of the clustering of extreme events [19] that our theory quantifies analytically in the rare event limit for Gaussian processes.

Generalization to active processes—Our theory is not restricted to passive processes but applies to any centered nonsmooth stationary Gaussian process with long-term memory. Indeed, our results (7), (10), (12), and (13) hold when one identifies $l^2 = \langle x^2\rangle_s$, while $l^* = l^2/L$ and $t^* = (l^*/\sqrt{\kappa})^{1/H}$, where κ and H are defined such that $\mathrm{var}[x(t)]_{x(0)=L} \simeq \kappa t^{2H}$ at short times. As an example of active process, we consider $x(t)$ satisfying

$$\dot x = -x + \xi(t), \qquad \langle\xi(t+\tau)\xi(t)\rangle = \frac{2H(2H-1)}{\tau^{2-2H}}, \qquad (14)$$

that displays long-term memory with $\alpha = 2-2H$ [50]. The values of κ, l, A for this process are given in SM [67].

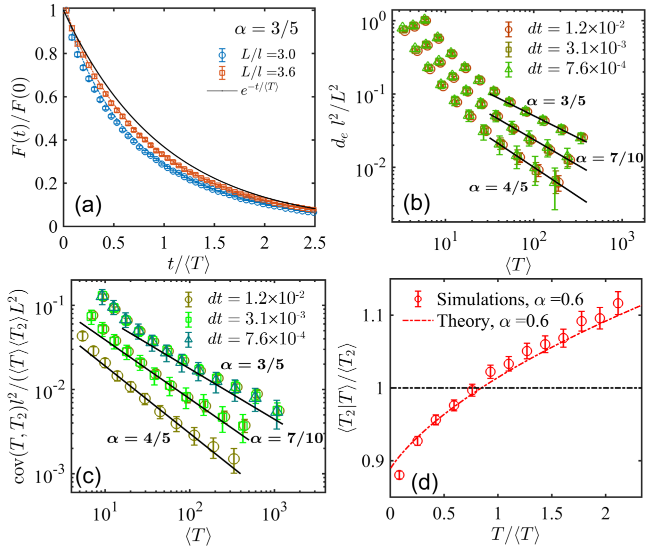

FIG. 4. Same quantities as in Figs. 2 and 3, but for the active process described by Eq. (14). (a) Distribution of FPTs, $F(t)$. (b) Parameter d_e quantifying the deviation to the exponential distribution. (c) Rescaled covariance between T and T_2. (d) Mean second passage conditioned on the first passage time. Symbols, simulations; lines, predictions; parameters, $\tau_w = 1$ [(c),(d)] and $L/l = 2.8$, $dt = 1.2 \times 10^{-2}$ [(d)].

Figure 4 shows that our formulas for the distribution of FPT and the coupling between the first and second passage times are in good agreement with simulations for this active process.

Conclusion—We have introduced a theoretical analysis showcasing the analytical link between the long-term memory properties and the clustering of extreme events [19]. On the basis of non-Markovian Gaussian stochastic processes with long-term memory, we have quantified the deviations between the distribution of first passage times to a rarely reached threshold to an exponential distribution, thus going beyond the Arrhenius paradigm. The theory also accounts for the joint statistics of first and second passage times, whose correlations are characterized by a small number of parameters. These parameters depend on the dynamics either close to the target or near the stable position, both of which are expected to be locally Gaussian even for non-Gaussian dynamics. We may thus expect that our theory could be generalized to non-Gaussian processes with long-term memory. In SM [67], for viscoelastic subdiffusion in an anharmonic potential (which is non-Gaussian), we show that correlations between successive passages exist and are well-described by our formulas. It would also be interesting to see if our theory could quantify such correlations for other models with slowly decaying memory of initial conditions, such as self-interacting processes [70,71].

Our method can be iterated to calculate the statistics of the n th passage time T_n, and the correlations between T_n and T_m. Our results could thus have application to the barrier crossing problem with viscoelastic memory, where each passage in the vicinity of the barrier can lead to a barrier crossing with a low probability. Our results are validated using extensive numerical simulations for both active and passive stochastic processes with long-term memory that are relevant in various physical areas (dynamics of polymers, proteins, tracers in viscoelastic fluids, etc.). Our findings should also be relevant for geophysical time series, which often display the long-term memory property [19,30]. Altogether, our results shed light, with analytical methods, on the correlations between successive occurrences of rare events due to long-term memory, beyond Arrhenius laws.

Acknowledgments—A. B. and T. G. acknowledge support of the grant “ComplexEncounters,” No. ANR-21-CE30-0020. T. G. thanks Olivier Bénichou, Raphael Voituriez, and Hamid Kellay for interesting discussions. Computer time for this study was provided by the computing facilities MCIA (Mesocentre de Calcul Intensif Aquitain) of the Université de Bordeaux and of the Université de Pau et des Pays de l’Adour.

Data availability—The data that support the findings of this article are openly available [72].

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End Matter

Details on the equations for the statistics of first passage times—Here, we outline the key steps that lead to the self-consistent equation that determines the function R. We start by considering Eq. (9) for $m = 1$, multiply this equation by x_1, and integrate the resulting equation with respect to x_1. This leads to

$$p(L,t)\mathbb{E}(x(t+t_1)|x(t)=L) = \int_0^t du F(t-u)\times$$
$$p(L,t|T=t-u)\mathbb{E}(x(t+t_1)|T=t-u; x(t)=L), \quad \text{(A1)}$$

where $\mathbb{E}(y|B;C)$ represent the conditional average of y given that the conditions B and C are realized. In particular, using existing expressions for conditional Gaussian variables [73], we have

$$\mathbb{E}(x(t+t_1)|x(t)=L) = L\frac{\sigma(t+t_1,t)}{\sigma(t,t)} = L\phi(t_1), \qquad \text{(A2)}$$

where $\sigma(t,t') = \mathrm{cov}[x(t),x(t')]$ denotes the covariance of the original process, and we have used $\sigma(t,t') = l^2\phi(|t-t'|)$ since the stochastic process $x(t)$ is stationary.

As mentioned in the main text, we evaluate the terms in Eq. (A1) with the assumption that the process in the future of the FPT is Gaussian, with a mean $\mu(t|\tau) \equiv \mathbb{E}[x(t)|T=\tau]$, and a covariance $\mathrm{cov}[x(t),x(t')|T=\tau] \equiv \sigma(t,t'|\tau)$ that is approximated as the covariance of the initial process given that $x(\tau)=L$ (not necessarily for the first time). Applying existing formulas for conditional Gaussian distributions [73], we can write

$$\sigma(t,t'|\tau) = \sigma(t,t') - \frac{\sigma(t,\tau)\sigma(t',\tau)}{\sigma(\tau,\tau)} = l^2[\phi(|t-t'|) - \phi(|\tau-t|)\phi(|\tau-t'|)], \quad \text{(A3)}$$

$$p(L,t|T=t-u) = \frac{e^{-[L-\mu(t|t-u)]^2/[2\sigma(t,t|t-u)]}}{\sqrt{2\pi\sigma(t,t|t-u)}}, \quad \text{(A4)}$$

$$\mathbb{E}[x(t+t_1)|T=t-u; x(t)=L] = \mu(t+t_1|t-u) - [\mu(t|t-u)-L]\frac{\sigma(t+t_1,t|t-u)}{\sigma(t,t|t-u)}. \quad \text{(A5)}$$

Equations (A1) and (3), together with (A4) and (A5), form a set of coupled integral equations that define μ and F.

We now analyze these equations in the rare event limit $L \to \infty$. First we consider the length and time scales, l^* and t^* respectively when trajectories are near to the threshold at $x=L$. For t of the order of T_{RE}, and at times $u \sim t^*$ after the FPT, we postulate that the displacements occur at the length scale l^*, so that

$$\mu(t|t-u) \simeq L - l^* f(U), \qquad U = u/t^*, \quad \text{(A6)}$$

where we have assumed that the trajectory after the FPT does not depend on the value of the FPT. Then, the PDF of $x=L$ at a time $u \sim t^*$ after the FPT is

$$p(L,t|T=t-u) \simeq \frac{q^*(u/t^*)}{l^*}, \qquad q^*(U) = \frac{e^{-\frac{f^2(U)}{2U^{2H}}}}{\sqrt{2\pi}U^H}. \quad \text{(A7)}$$

We refer to Ref. [52] for the derivation of an equation for f, which is involved in the calculation of the mean FPT (in the absence of long-term memory).

Next, for larger times, when $\bar{t} = t/T_{\mathrm{RE}}$ and $\bar{u} = u/T_{\mathrm{RE}}$ are of order 1, we use the ansatz

$$\mu(t|t-u) \simeq \frac{LA}{T_{\mathrm{RE}}^{\alpha}} R(\bar{u}, \bar{t}-\bar{u}), \qquad [t,u = \mathcal{O}(T_{\mathrm{RE}})], \quad \text{(A8)}$$

which is compatible with Eq. (5) [see Eq. (A4) with $\sigma(t,t|t-u) \simeq l^2 + \mathcal{O}(\varepsilon^2 l^4/(L^4 \bar{t}^{2\alpha})) \simeq l^2$]. Now, we estimate the terms appearing in Eq. (A1) at different time-scales, when $t, t_1 \sim \mathcal{O}(T_{\mathrm{RE}})$. First, when $u \sim t^*$, evaluating (A5) with the scaling forms (A6) and (A8) for μ and the fact that $u \ll t_1$, we obtain

$$\begin{aligned}\mathbb{E}[x(t+t_1)|T=t-u; x(t)=L] &\simeq \frac{LA}{T_{\mathrm{RE}}^{\alpha}} R(\overline{t_1}, \bar{t}) + l^* f\left(\frac{u}{t^*}\right) \frac{\phi(t_1) - \phi(t_1)\phi(u)}{1-\phi^2(u)} \\ &\simeq \frac{LA}{T_{\mathrm{RE}}^{\alpha}} R(\overline{t_1}, \bar{t}),\end{aligned} \quad \text{(A9)}$$

where in the second equality we have used that $\phi(u) \simeq 1$, $\phi(t_1) \simeq A/t_1^{\alpha}$, and $l^* \ll L$. Next, we evaluate the same quantity for $u \sim \mathcal{O}(T_{\mathrm{RE}})$, still with $t, t_1 = \mathcal{O}(T_{\mathrm{RE}})$, which leads to

$$\begin{aligned}\mathbb{E}[x(t+t_1)|T=t-u; x(t)=L] &\simeq \frac{LA}{T_{\mathrm{RE}}^{\alpha}} R(\overline{t_1}+\bar{u}, \bar{t}-\bar{u}) - \left(\frac{LA}{T_{\mathrm{RE}}^{\alpha}} R(\bar{u}, \bar{t}-\bar{u}) - L\right)\phi(t_1) \\ &\simeq \frac{LA}{T_{\mathrm{RE}}^{\alpha}} \left[R(\overline{t_1}+\bar{u}, \bar{t}-\bar{u}) + \frac{1}{\overline{t_1}^{\alpha}}\right],\end{aligned} \quad \text{(A10)}$$

where in the last equality we have used $\phi(t_1) \simeq A/t_1^{\alpha}$.

We can now evaluate the integral in Eq. (A1), where we take into account a contribution with $u = \mathcal{O}(t^*)$ [where we use (A9) and (A7)] and the contribution of $u = \mathcal{O}(T_{\mathrm{RE}})$ [where we use (A10) and $p(L,t|T=t-u) \simeq p_s(L)$], we obtain at lowest order in ε,

$$\frac{1}{\bar{t}_1^{\alpha}} = e^{-\bar{t}} R(\bar{t}_1, \bar{t}) + \int_0^{\bar{t}} d\bar{u} \left(R(\bar{t}_1+\bar{u}, \bar{t}-\bar{u}) + \frac{1}{\bar{t}_1^{\alpha}}\right) e^{\bar{u}-\bar{t}}. \quad \text{(A11)}$$

This self-consistent equation for R can be considerably simplified if one introduces $W(\bar{t}_1, y) = e^{-\bar{t}} R(\bar{t}_1, t)$, with $y = \bar{t} + \bar{t}_1$, and $v = \bar{u} + \bar{t}_1$ (inside the integral), leading to

$$W(\bar{t}_1, y) + \int_{\bar{t}_1}^{y} dv W(v, y) = \frac{e^{-y+\bar{t}_1}}{\bar{t}_1^{\alpha}}. \quad \text{(A12)}$$

This Volterra integral equation of the second kind for $W(\cdot, y)$ admits the known solution [74]

$$W(\bar{t}_1, y) = \frac{e^{-y+\bar{t}_1}}{\bar{t}_1^{\alpha}} + \int_y^{\bar{t}_1} dv e^{\bar{t}_1 - v} \frac{e^{-y+v}}{v^{\alpha}}. \quad \text{(A13)}$$

Simplifying the integral and expressing the result for the quantity R leads to

$$R(\bar{t}, \bar{u}) = \frac{\bar{t}^{-\alpha}(1-\alpha+\bar{t}) - (\bar{t}+\bar{u})^{1-\alpha}}{1-\alpha}. \quad \text{(A14)}$$

With this value of R, the equation (8) for F_1 can be readily solved upon using the Laplace transform, leading to the final solution (10).

Details on the equations for the statistics of second passage times—We define $F_2(\tau_2|\tau)$ as the PDF of the second passage time T_2 conditional to $T=\tau$. Let us define the stochastic process after the FPT, given that $T=\tau$, as $y(t|\tau)\equiv x[t-(\tau_w+\tau)]$, for all trajectories with $T=\tau$. Our strategy is to apply the methods of the previous section to this process, which is still Gaussian, but noncentered. In particular, at the scale T_{RE}, its mean reads

$$\langle y(t|\tau)\rangle \simeq \frac{LA}{T_{\mathrm{RE}}^{\alpha}}R(\bar{t}+\overline{\tau_w},\bar{\tau})\simeq \frac{LA}{T_{\mathrm{RE}}^{\alpha}}R(\bar{t},\bar{\tau}), \qquad \text{(B1)}$$

where $\bar{\tau}_w=\tau_w/T_{\mathrm{RE}}\ll 1$. For its covariance, one has $\mathrm{cov}[y(t|\tau),y(t'|\tau)]\simeq\sigma(t+\tau,t'+\tau|\tau)$. Importantly, at the scale $t,t'=\mathcal{O}(T_{\mathrm{RE}})$, $\mathrm{cov}[y(t|\tau),y(t'|\tau)]\simeq l^2\phi(|t-t'|)$ is the same as that of the original process $x(t)$.

Next, let us note $p_2(y,t)$ the PDF of $y(t|\tau)$ (here we omit the fixed quantity τ in p_2). Using the above expressions for the mean and variance of y and formulas for conditional Gaussian distributions [73], we obtain

$$p_2(L,t)\simeq p_s(L)(1+\varepsilon R(\bar{t},\bar{\tau})), \qquad \text{(B2)}$$

$$\mathbb{E}(y(t+t_1|\tau)|y(t|\tau)=L)\simeq\frac{LA}{T_{\mathrm{RE}}^{\alpha}}\left(R(\bar{t}+\bar{t}_1,\bar{\tau})+\frac{1}{\bar{t}_1^{\alpha}}\right). \qquad \text{(B3)}$$

We anticipate that the distribution of second passage times is exponential in the absence of long-term memory, so that we look for solutions under the form

$$F_2(\tau_2|\tau)=\frac{1}{T_{\mathrm{RE}}}[e^{-\bar{\tau}_2}+\varepsilon h_2(\bar{\tau}_2|\bar{\tau})+\cdots], \qquad \text{(B4)}$$

where $\bar{\tau}_2=\tau_2/T_{\mathrm{RE}}$. We also work with the assumption that the trajectories after the second passage, given that the second passage is τ_2, are Gaussian distributed, with a covariance approximated by the covariance of y conditional to $y(\tau_2|\tau)=L$, and a mean which, at the timescale T_{RE}, behaves as

$$\mathbb{E}[y(t|\tau)|T_2=t-u]=\frac{LA}{T_{\mathrm{RE}}^{\alpha}}R_2(\bar{u},\bar{t}-\bar{u}|\bar{\tau}). \qquad \text{(B5)}$$

Also, we assume that, at the timescale t^*, the statistics of paths after the second passage does not depend on the value of the second passage, and it is the same as the one after the first passage. As a consequence, we can use the same equations (3) and (A1) as for the analysis of $x(t)$ if we replace F by F_2, p by p_2, x by y, and R by R_2. The only differences come from the evaluation of $p_2(L,t)$ and the conditional value of $\langle y(t+t_1)\rangle$ which are given by (B2) and (B3). Applying the procedure of the previous section we finally obtain

$$\begin{aligned}&R(\bar{t}+\bar{t}_1,\bar{\tau})+\frac{1}{\bar{t}_1^{\alpha}}\\ &=e^{-\bar{t}}R_2(\bar{t}_1,\bar{t}|\bar{\tau})+\int_0^{\bar{t}}d\bar{u}e^{-(\bar{t}-\bar{u})}\left[R_2(\bar{t}_1+\bar{u},\bar{t}-\bar{u}|\bar{\tau})+\frac{1}{\bar{t}_1^{\alpha}}\right],\end{aligned} \qquad \text{(B6)}$$

$$\begin{aligned}R(\bar{t},\bar{\tau})=&\,h_2(\bar{t}|\bar{\tau})+\int_0^{\bar{t}}d\bar{u}h_2(\bar{t}-\bar{u}|\bar{\tau})\\ &+\int_0^{\bar{t}}d\bar{u}e^{-(\bar{t}-\bar{u})}R_2(\bar{u},\bar{t}-\bar{u}|\bar{\tau}).\end{aligned} \qquad \text{(B7)}$$

Note that these equations for R_2, h_2 are the same as the equations for R,F_1 [compare with Eqs. (8) and (A11)], except for additional terms in the left-hand sides, which come from Eqs. (B2) and (B3). These equations can be solved by standard methods described in SM [67], leading to explicit values for R_2 and h_2 and then, since one knows the distribution F, to an explicit expression for the joint distribution of T,T_2.