How Rare Events Remember

A new theory of rare recurrent events dispenses with the simplifying assumption that recent events lack memory of previous ones.

By **Nicolas Levernier**

Recurring earthquakes, stock-market crashes, catastrophic floods, and other rare events are statistically unlikely but significantly impactful. Their prediction is commonly based on the Arrhenius-Kramers paradigm [1], one of the most broadly applied frameworks in statistical physics. Implicit in this formalism are two strong universal features. First, the distribution of times to reach a rare event is exponential and independent of initial conditions, implying no correlation exists between successive rare events. Second, the mean waiting time increases exponentially with the size of the energy barrier (or effective energy) to be overcome. Despite its applicability, nature sometimes defies the Arrhenius-Kramers paradigm. Proteins can cross energy barriers with nonexponential kinetics [2], rainfall extremes can cluster in time [3] (Fig. 1), and seismic events can display correlations [4]. Although long-term memory has often been proposed as a possible explanation for nature's recalcitrance, a theoretical framework for investigating it has remained lacking—until now. Apurba Biswas and Thomas Guérin of the University of Bordeaux in France have provided the first fully analytical theory [5]. Their work quantifies how long-term memory in a stochastic process induces correlations between successive rare events and drives the waiting-time distribution away from its assumed exponential form. The result constitutes a substantial step beyond the Arrhenius-Kramers paradigm and toward a predictive theory of extreme events.

Figure 1: Cars piled up in Ellicott City, Maryland, in the aftermath of a May 2018 storm that dropped 20 cm of rain in two hours. In July 2016, a storm dropped 15 cm of rain in two hours on the same town. **Credit: Howard County Government**

The Arrhenius-Kramers paradigm rests on a process with a clean separation of timescales. In the weak-noise limit, the mean time needed to escape from a potential well grows exponentially with the well's height. It eventually exceeds the time needed for the system to relax to its initial state, so that subsequent perturbations become statistically independent. This universality fails for processes with long-term memory. Here, autocorrelations decay over time as a power law rather than exponentially, making the relaxation time effectively infinite. Such processes arise across an extraordinary range of phenomena—among them, polymer and protein dynamics [6], tracer transport in viscoelastic fluids, and flood recurrence [7].

A key quantity in the study of rare events is first-passage time, which is the time a stochastic process takes to reach a threshold for the first time. The threshold could be a spatial boundary, the top of an energy barrier, or a final state. Many studies have attempted to unveil the consequences of long-term memory on first-passage time. Most have relied on one kind of approximation or another. Previous work by Guérin and collaborators computed mean first-passage times for non-Markovian walkers—that is, systems whose future trajectories depend on past trajectories [8]. However, the full

DOI: 10.1103/Physics.19.105

distribution of first-passage times—and crucially the correlations between successive passages—remained out of reach.

In their most recent work, Biswas and Guérin considered a process described by a generalized Langevin equation—a stochastic differential equation describing how noise and past states of the system influence its current motion. Biswas and Guérin's equation includes a source of correlated noise, whose correlations decay as a power law, and a force arising from a harmonic-well potential. Having set up their equation, they asked themselves, What are the statistics of the first- and second-passage times of the level L, which is large compared with the typical level l that the system explores in its steady state?

The key advance that Biswas and Guérin made is to recognize and exploit the extreme separation between two timescales: the small local timescale governing dynamics near the threshold and the exponentially large mean passage time T to reach the threshold. Biswas and Guérin started from an exact identity that partitions the probability of observing the process at the threshold over all possible values of the first-passage time. Next, they expanded the distribution of first-passage times in powers of a small parameter $\varepsilon = AL^2/l^2 T^{\alpha}$. Here, A and α come from the power-law decay of the autocorrelations, which goes as $At^{-\alpha}$.

The crucial point is that the noise's long-term correlations cause the relaxation of the stochastic process to decay as $t^{-\alpha}$ rather than as an exponential. This slow relaxation, invisible when future events have no memory of past ones, contaminates the statistics of future passages, and its importance is quantified by ε. At zeroth order in ε, the theory correctly recovers the exponential distribution and the Arrhenius-Kramers mean passage time, consistent with earlier mathematical results [**9**]. At first order, it yields an explicit nonexponential correction to the first-passage-time distribution, along with an expression for the covariance between the first- and second-passage times. This last formula captures the central phenomenon of event clustering: If a first passage took unusually long, the hidden degrees of freedom responsible for long memory will maintain an elevated effective barrier, making the next passage likely to be long as well—and vice versa for an unusually short first-passage time.

What makes the work of Biswas and Guérin particularly compelling is that it delivers something long sought: a first-principles, parameter-free prediction of the full distribution of first-passage times and of the interevent correlations in a non-Markovian system. As inputs, it requires only α and A, both of which can be derived from measurements made when the system is in equilibrium. This thorough analysis of successive rare events will surely have direct consequences for risk assessment in climate science, seismology, and finance. The framework also provides single-molecule experimentalists with a concrete, parameter-free prediction to test: The covariance between successive barrier-crossing times in a protein or a polymer should follow the power-law scaling derived by Biswas and Guérin.

The statistical study of rare events has long relied on the assumption that events are Markovian—that is, they don't depend on past events. The assumption was the unavoidable price of analytical tractability. Biswas and Guérin's work makes a compelling case that the non-Markovian world is within reach.

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REFERENCES

1. H.A. Kramers, "Brownian motion in a field of force and the diffusion model of chemical reactions," **Physica 7, 284 (1940);** P. Hänggi *et al.*, "Reaction-rate theory: Fifty years after Kramers," **Rev. Mod. Phys. 62, 251 (1990).**
2. H. P. Lu *et al.*, "Single-molecule enzymatic dynamics," **Science 282, 1877 (1998);** O. Flomenbom *et al.*, "Stretched exponential decay and correlations in the catalytic activity of fluctuating single lipase molecules," **Proc. Natl. Acad. Sci. U.S.A. 102, 2368 (2005).**
3. A. Bunde *et al.*, "Long-term memory: A natural mechanism for the clustering of extreme events and anomalous residual times in climate records," **Phys. Rev. Lett. 94, 048701 (2005).**
4. S. R. Carpenter *et al.*, "Long-range dependence and extreme values of precipitation, phosphorus load, and Cyanobacteria," **Proc. Natl. Acad. Sci. U.S.A. 119, e2214343119 (2022).**
5. A. Biswas and T. Guérin, "Correlations between rare events for Gaussian stochastic processes with long-term memory," **Phys. Rev. Lett. 137, 077101 (2026).**
6. S. C. Kou and X. S. Xie, "Generalized Langevin equation with fractional Gaussian noise: Subdiffusion within a single protein

molecule," **Phys. Rev. Lett. 93, 180603 (2004);** W. Min *et al.*, "Observation of a power-law memory kernel for fluctuations within a single protein molecule," **Phys. Rev. Lett. 94, 198302 (2005).**

7. B. B. Mandelbrot and J. R. Wallis, "Noah, Joseph, and operational hydrology," **Water Resour. Res. 4, 909 (1968);** S. Lennartz *et al.*, "Long-term memory in earthquakes and the distribution of interoccurrence times," **Europhys. Lett. 81, 69001 (2008).**
8. T. Guérin *et al.*, "Mean first-passage times of non-Markovian random walkers in confinement," **Nature 534, 356 (2016);** N. Levernier *et al.*, "Kinetics of rare events for non-Markovian stationary processes and application to polymer dynamics," **Phys. Rev. Res. 2, 012057 (2020).**
9. J. Pickands III, "Efficient estimation of a probability density function," **Ann. Math. Statist. 40, 854 (1969).**